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Instantons ~~Quiver varieties~~ and double affine Grassmannian

⊗ Review of geometric Satake correspondence

G : reductive grp $/\mathbb{C}$

$K = \mathbb{C}((s)) \supset \mathcal{O} = \mathbb{C}[[s]]$

$Gr_G = G(K)/G(\mathcal{O})$: affine Grassmannian

- ∞-dimensional variety
- $Gr_G \underset{\text{homotopic}}{\simeq} \Omega G_{cpt}$ based loops

$G(\mathcal{O})$-orbits on Gr_G

$\leftrightarrow \lambda \in \Lambda^+$: dominant coweights

$$\left(\begin{array}{l} \Lambda^+ \subset \Lambda = \text{coweight lattice of } G = \mathrm{Hom}(\mathbb{G}_m, \underset{\substack{\uparrow \\ \text{maximal torus}}}{T}) \subset G(K) \\ \qquad \cong \text{weight lattice of } {}^LG \\ \lambda \in \Lambda^+ \leftrightarrow \text{dominant weight of } {}^LG \leftrightarrow \text{f.d. irr. rep } V(\lambda) \text{ of } {}^LG \end{array} \right.$$

$$Gr_G = \bigsqcup_{\lambda \in \Lambda^+} Gr_G^\lambda$$: stratification (analog of Schubert cells)

closure $\overline{Gr_G^\lambda}$: finite dimensional projective variety usually singular

$IC(\overline{Gr_G^\lambda})$: intersection cohomology complex of $\overline{Gr_G^\lambda}$ (Goresky–MacPherson)
(not sheaf, cpx of constructible sheaves)

(extend $\mathbb{C}_{Gr_G^\lambda}$ rather nontrivial way to $\overline{Gr_G^\lambda}$ so that Poincaré duality holds

$\mathcal{P} = \mathrm{Perv}_{G(\mathcal{O})} Gr_G$: abelian category of $G(\mathcal{O})$-equiv perv. sheaves on Gr

It has a tensor structure via "convolution diagram"

$$G(K) \times_{G(\mathcal{O})} Gr_G = Gr_G \tilde{\times} Gr_G \xrightarrow{\varpi} Gr_G$$

Gr_G-b'dle over Gr_G $\qquad A * B := \varpi_*(A \tilde{\boxtimes} B)$

Th. (Lusztig, Ginzburg, Beilinson–Drinfeld, Mirković–Vilonen)

$(\mathcal{P}, *) \cong (\mathrm{Rep}(G^\vee), \otimes)$ as $\otimes$-categories

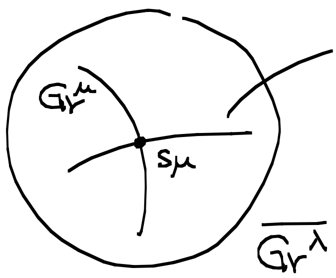
s.t. $\mathbb{H}^*(IC(\overline{Gr_G^\lambda})) \cong V(\lambda)$
highest weight representation

How about weight space ?

$V(\lambda)_\mu$: weight space $=$ stalk of $IC(\overline{Gr_G^\lambda})$ at $s^\mu \in Gr_G^\mu$

This is the starting point of the geometric Langlands.

- more suitable for double affine generalization



$\overline{M}^\lambda_\mu$: transversal slice to Gr_G^μ in $\overline{Gr_G^\lambda}$

$$\overline{M}^\lambda_\mu \cap Gr_G^\lambda =: M^\lambda_\mu \underset{\text{open}}{\subset} \overline{M}^\lambda_\mu$$

$$V(\lambda)_\mu \cong IC(\overline{Gr^\lambda}) \text{ at } s^\mu \cong IC(\overline{M}^\lambda_\mu) \text{ at } s^\mu$$

<u>Question</u>

What is the affine analog of the affine Grassmann
= double affine Grassmann ?

$V(\lambda)$: ∞-dimensional

$V(\lambda) \otimes V(\mu)$: ∞-direct sum of $V(\nu)$'s

Consider only integrable highest weight rep.

$\Rightarrow$ controllable ∞ sum !

But geometric side: $Gr_{G_{aff}} = G_{aff}(K)/G_{aff}(\mathcal{O})$

and orbits are highly ∞-dimensional!

difficult to define IC sheaves ......

Proposal (Braverman - Finkelberg 0711.2083)

analog of $\overline{M}^{\lambda}_{\mu}$ = Uhlenbeck partial cptification of G-instantons on $\mathbb{R}^4/\mathbb{Z}_\ell$

ℓ = level of the rep. of aff. KM group

- $H^*(IC(\text{analog of } \overline{M}^{\lambda}_{\mu})) \cong V(\lambda)_\mu$ rep. of $(G_{aff})^\vee$
- certain diagram $\dashleftarrow\dashrightarrow$ $\otimes$
  explained later

G : simple & simply-connected

$\mathrm{Bun}_G^k(\mathbb{C}^2)$ = framed moduli space of G_{cpt}-instantons on S^4 with $c_2 = k$ trivialization at ∞

= framed moduli space of algebraic G-bdles on $\mathbb{CP}^2$ trivialization at $l_\infty \subset \mathbb{CP}^2$

smooth of dim $= 2kh^\vee$

$$\mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2) := \bigsqcup_{0 \leq k' \leq k} \mathrm{Bun}_G^{k'}(\mathbb{C}^2) \times S^{k-k'}\mathbb{C}^2$$

Uhlenbeck partial cptification

Fix a hom $\mu : \mathbb{Z}_l \to G$

$$\begin{array}{c} \mathbb{Z}_l \\ \cap \\ SL(2) \subset GL(2) \end{array}$$

$$\mathbb{Z}_l \curvearrowright \mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2)$$

through the action of diagonal emb. to $(\mathrm{incl} \times \mu) : \mathbb{Z}_l \to GL(2) \times G$

fixed pts $=: \mathrm{Bun}_G^k(\mathbb{C}^2/\mathbb{Z}_l)$

another inv. $\lambda : \mathbb{Z}_\ell \longrightarrow G$ hom.
action corr. to
the fiber at $0 \in \mathbb{C}^2$

$\mathcal{U}_{G,\mu}^{\lambda,k}$:= fixed pt set

<u>Technical conjecture</u>
$\mathcal{U}_{G,\mu}^{\lambda,k}$: irreducible

<u>Lemma</u> (BF)
$\lambda, \mu \in \mathrm{Hom}(\mathbb{Z}_\ell, G)/_{\text{conj.}} \overset{\text{bijection}}{\longleftrightarrow}$ level ℓ wts of $\hat{\mathfrak{g}}^\vee$

$\hat{\mathfrak{g}}^\vee$ does not contain the degree operator d

lifts to $(\mathfrak{g}_{\mathrm{aff}})^\vee$: unique up to $\mathbb{C}\delta$

$\tilde{\lambda}, \tilde{\mu}$: lifts s.t. $\langle \tilde{\lambda} - \tilde{\mu}, d \rangle = \underset{\substack{\| \\ c_2}}{k}\ell$

<u>Main Conjecture 1</u>

$$H^*(IC(\mathcal{U}_{G,\mu}^{\lambda,d})) = V(\tilde{\lambda})_{\tilde{\mu}}$$

<u>Rem</u>

① $V(\tilde{\lambda}+c\delta)_{\tilde{\mu}+c\delta} \cong V(\tilde{\lambda})_{\tilde{\mu}}$

② $\exists$ graded version

LHS : cohomological grading

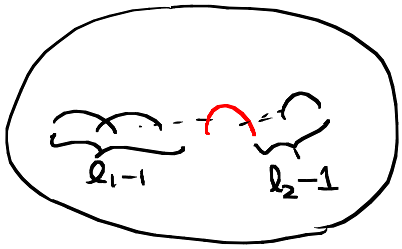
RHS : principal nilpotent

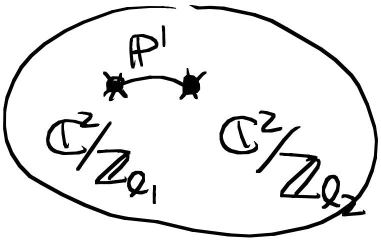
Tensor product

$\ell = \ell_1 + \ell_2$

partial resolution

$$\mathbb{C}^2/\mathbb{Z}_\ell \longleftarrow X_{\ell_1,\ell_2} \longleftarrow$$





$(\ell-1)$ $\mathbb{P}^1$'s

$\|$

$(\ell_1-1)+(\ell_2-1)+1$

Consider Uhlenbeck space on X_{ℓ_1,ℓ_2}

$\mathcal{U}^{\lambda_1,\lambda_2,d}_{G,\mu}$ $\qquad$ $\lambda_1, \lambda_2 : \mathbb{Z}/\ell_1, \mathbb{Z}/\ell_2 \to G$

level ℓ_1, ℓ_2 weights

Technical conjecture

$\exists$ semismall morphism $\pi : \mathcal{U}_{G,\mu}^{\lambda_1,\lambda_2,d}(X_{\ell_1,\ell_2}) \to \mathcal{U}_{G,\mu}^{\lambda,d}(\mathbb{C}^2)$, $\lambda = \lambda_1+\lambda_2$

Main Conjecture 2

$$\pi_* IC(\mathcal{U}_{G,\mu}^{\lambda_1,\lambda_2,d}(X_{\ell_1,\ell_2})) = \bigoplus IC(\mathcal{U}_{G,\mu}^{\lambda',d})^{\oplus m_{\lambda_1,\lambda_2}^{\lambda'}}$$

$\oplus$ other

with $\left(V(\lambda_1)\otimes V(\lambda_2)\right)_\mu = \bigoplus_{\lambda'} V(\lambda')_\mu^{\oplus m_{\lambda_1,\lambda_2}^{\lambda'}}$

Th. conjectures (except graded version of MC1) are true for $G = SL(r)$

$G = SL(r)$ ---- $\mathcal{U}_{G,\mu}^{\lambda,d}$ is an (affine) quiver variety

its IC sheaf was computed

---- related to rep. theory of $\widehat{\mathfrak{sl}}_\ell$ at level $= r$

I. Frenkel level-rk duality

$$\widehat{sl}(r)_\ell \longleftrightarrow \widehat{sl}(\ell)_r$$

$\otimes$ $\longleftrightarrow$ branching to $\widehat{sl}(\ell_1)_r \oplus \widehat{sl}(\ell_2)_r$

I develop the theory for the branching in the quiver variety

<u>Rem</u> ① technical advantage for $G = SL(r)$
$\exists$ nice resolution of $\mathcal{U}^{\wedge}_{G,\mu}$

(Gieseker cptification)

② quiver variety generalization to other $\Gamma \subset SL(2)$
$\leftrightarrow$ affine ADE

But the gauge group is $SL(r)$ or $GL(r)$

<u>Question</u>. What kind of algebraic structure controls e.g. G_{E_8} - instantons on $\mathbb{R}^4/\Gamma_{E_8}$?

I. Frenkel's joke: Monster ?